Abstract

We consider the solution of large sparse nonsymmetric generalized eigenvalue problems. An implicitly restarted refined shift-and-invert Arnoldi method is proposed that cater for the possible non-convergence of approximate eigenvectors when using the shift-and-invert scheme of Sorensen. Comparisons are drawn on the refined method and the implicitly restarted shift-and-invert algorithm, showing the better performance of the former. We derive residual bounds for the two variants of shift-and-invert Arnoldi methods and numerical results validate the proposed bounds. Thus the algorithms are provided with necessary theoretical background.

Keywords  Nonsymmetric matrix, Shift-and-Invert Arnoldi’s method, refined, implicit, Tchebychev polynomial, shifts

1. Introduction

Large non-symmetric generalized eigenvalue problems

\[ Ax = \lambda B x, \]  \hspace{1cm} (1)

where A and B are matrices of order \( n \), arise in many scientific applications (e.g. Bai et al. (1995) and references therein). We are interested in finding a few interior eigenvalues of \( A \) close to a given shift \( \sigma \), and the associated eigenvectors of the matrix pair \( (A, B) \). The shift-and-invert Arnoldi algorithm of Saad (1989) is a commonly used technique for this type of large problem, where it is assumed that the LU decomposition of the invertible matrix \( A - \sigma B \) is appropriate and can be performed cheaply. However, convergence of the approximate eigenvectors are not guaranteed even if the Ritz values do. Jia and Zhang (2002) described a refined shift-and-invert Arnoldi algorithm to correct this drawback, where it is shown that the refined Ritz vectors are more efficient.

In (Lehoucq et al. 1998, Sorensen 1992) the shift-and-invert Arnoldi method is restarted implicitly (IRSIA) by using the QR algorithm with many advantages, in which one can be a fixed storage requirement. However this implicit scheme has the same convergence problem as the shift-and-invert Arnoldi algorithm. This motivates us to describe an implicitly restarted refined shift-and-invert Arnoldi method (IRRSIA) by using the same arguments as in the work of Jia (1999). Numerical experiments show the much more efficiency of the proposed methods compared to that of the implicitly restarted shift-and-invert Arnoldi algorithm.

In the second part of the paper, we are concerned with residuals bounds of the shift-and-invert implicit scheme and its refined versions. Many study on the convergence of the Krylov subspace to an invariant subspace
have been carried out by robustly measuring the angle between them. Saad (1980) derived bound between a single eigenvector and the Krylov subspace for real and simple matrix. Jia (1995) generalized this bound for defective and nonderogatory matrix, but uses the Jordan structure of the coefficient matrix and derivatives of the approximating polynomial. In (Beattie et al. 2004), new bound for single-vector Krylov subspace algorithms have been established that impose no limit on the size of the wanted invariant subspace. Similar bound was proposed in (Beattie et al. 2005) that handle the complexities of non-symmetric matrices, and this bound showed that the instability of undesired eigenvalues have a great impact on convergence as compared to the ill-conditioning of the wanted eigenvalues.

However, there are few residual bounds. Saad (1980) presented a residual bound for the Arnoldi method where its convergence is related to the Tchebychev polynomial of first kind. Jia (1997) expresses the residual bound in terms of the condition number of the eigenvector matrix and a polynomial approximation. Also in (Jia and Zhang 2002), residual bounds for both explicit shift-and-invert Arnoldi method and its refined version are described. Hence, this motivates us to established new theoretical residual bounds for the generalized eigenvalue problems using approximation by polynomial.

2. Results

Let \( \text{cpu} \) denotes the cpu timings in seconds, \( \text{mv} \) is the number of matrix-vector products with the matrix \( C \) and
\[
\rho_i = \| (A - \lambda_i(t \omega) B) x_i(t \omega) \| \quad \text{for IRSIA and } \rho_i = \| (A - \lambda_i(t \omega) B) u_i(t \omega) \| \quad \text{for IRRSIA.}
\]

We compare the IRSIA and IRRSIA algorithms by considering the generalized eigenvalue problem that arises from the analysis of dissipative magnetohydrodynamic (MHD). The physical aim of these MHD systems that combines the Maxwell’s and fluid equations, is to obtain nuclear energy from the fusion of light nuclei. For further details see Bai et al. (1995). The generalized eigenproblem results when the Galerkin method in conjunction with finite elements is applied to the MHD equations.

The matrix pair \((A, B)\) with order 416 is taken from Bai et al. (1995), where we computed six eigenvalues of the problem around different shifts \( \sigma \). Table 1 reports the results obtained. We observe that IRRSIA requires lesser amount of matrix-vector products and cpu timings than IRSIA.

### 2.1 Validation of the Proposed Bounds

We demonstrate the proposed convergence bounds by using the matrix pair \((A, B)\) of order 416 from the MHD systems. We choose \( \sigma = -1 \) and compute two eigenpairs using a Krylov subspace of dimension 5 for both IRSIA and IRRSIA methods with a tolerance \( 10^{-6} \). Fig. 1 shows that the derived bounds are validated numerically. However, note that convergence of the second eigenpair is not attained using IRSIA scheme.
Figure 1. Illustration for $\sigma = -1$. In (a)-(d) the residual norm is plotted against matrix-vector products, where the dotted line represents the actual residual norm, and the dashed, solid and dashed-dotted lines represent the respective bounds.

3. Conclusion

We considered implicitly restarted techniques for generalized eigenvalue problem. We described a new scheme namely the implicitly restarted refined shift-and-invert that can outperform the implicit algorithm of Sorensen. It seemed that all the numerical results have shown that our proposed algorithms were often far superior. We also derived some theoretical residual bounds that are different from existing ones and numerical results validate the bounds.

References


